key

Math 19 Problem Set 12: p264-265 Ex. 1acef, 2,3

1.(a) $h(0)=0$ & $h(R)=0$

$\boxed{c>0}$: $h(x)=\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}$

$h(0)=\alpha+\beta=0$

$\alpha=-\beta$

$h(R)=\alpha e^{\sqrt{c}R}+\beta e^{-\sqrt{c}R}$

$=\alpha(e^{\sqrt{c}R}-e^{-\sqrt{c}R})=0$

$\boxed{\alpha=0,\ \beta=0}$

$\boxed{c=0}$: $h(x)=\alpha+\beta x$

$h(0)=\boxed{\alpha=0}$

$h(R)=\beta R=0$ $\boxed{\beta=0}$

$\boxed{c<0}$: $h(x)=\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)$

$h(0)=\boxed{\alpha=0}$

$h(R)=\beta\sin(\sqrt{-c}R)=0$

$\boxed{\beta=0}$ or $\sqrt{-c}R=n\pi$

$\boxed{\beta\in\mathbb{R},\ c=-\left(\frac{n\pi}{R}\right)^2}$

(c) $\frac{dh}{dx}(0)=0$ & $h(R)=0$

$\boxed{c>0}$: $h(x)=\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}$

$h'(x)=\sqrt{c}\alpha e^{\sqrt{c}x}-\sqrt{c}\beta e^{-\sqrt{c}x}$

$h'(0)=\sqrt{c}(\alpha-\beta)=0$

$\alpha=\beta$

$h(R)=\alpha(e^{\sqrt{c}x}+e^{-\sqrt{c}x})=0$

$\boxed{\alpha=0\quad\beta=0}$

$\boxed{c=0}$: $h(x)=\alpha+\beta x$

$h'(x)=\beta$

$h'(0)=\boxed{\beta=0}$

$h(R)=\boxed{\alpha=0}$

$\boxed{c<0}$: $h(x)=\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)$

$h'(x)=-\alpha\sqrt{-c}\sin(\sqrt{-c}x)+\beta\sqrt{-c}\cos(\sqrt{-c}x)$

$h'(0)=\beta\sqrt{-c}=0$ $\boxed{\beta=0}$

$h(R)=\alpha\cos(\sqrt{-c}R)=0$

$\boxed{\alpha=0 \text{ or } c=-\left(\frac{(n+\frac{1}{2})\pi}{R}\right)^2,\alpha\in\mathbb{R}}$

(e) $h(0)=0$, & $h(R)=1$

$\boxed{c>0}$: $h(x)=\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}$

$h(0)=\alpha+\beta=0$ $\quad\alpha=-\beta$

$h(R)=\alpha(e^{\sqrt{c}R}-e^{-\sqrt{c}R})=1$

$\boxed{\alpha=(e^{\sqrt{c}R}-e^{-\sqrt{c}R})^{-1}\quad\beta=-(e^{\sqrt{c}R}-e^{-\sqrt{c}R})^{-1}}$

$\boxed{c=0}$: $h(x)=\alpha+\beta x$

$h(0)=\boxed{\alpha=0}$

$h(R)=\beta R=1$ $\quad\boxed{\beta=\frac{1}{R}}$

$\boxed{c<0}$: $h(x)=\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)$

$h(0)=\boxed{\alpha=0}$

$h(R)=\beta\sin(\sqrt{-c}R)=1$

$\boxed{\beta=\frac{1}{\sin(\sqrt{-c}R)}}$

(f) $h(0)=-1$ & $h(R)=1$

$\boxed{c>0}$: $h(x)=\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}$

$h(0)=\alpha+\beta=-1$ $\quad\alpha=-1-\beta$

$h(R)=(-1-\beta)e^{\sqrt{c}R}+\beta e^{-\sqrt{c}R}=1$

$\boxed{\beta=\frac{1+e^{\sqrt{c}R}}{(-e^{\sqrt{c}R}+e^{-\sqrt{c}R})}\quad\alpha=-1-\frac{1+e^{\sqrt{c}R}}{e^{\sqrt{c}R}-e^{\sqrt{c}R}}}$

$\boxed{c=0}$: $h(x)=\alpha+\beta x$

$h(0)=\boxed{\alpha=-1}$

$h(R)=-1+\beta R=1$

$\boxed{\beta=\frac{2}{R}}$

$\boxed{c<0}$: $h(x)=\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)$

$h(0)=\boxed{\alpha=-1}$

$h(R)=-\cos(\sqrt{-c}R)+\beta\sin(\sqrt{-c}R)=1$

$\boxed{\beta=\frac{1+\cos(\sqrt{-c}R)}{\sin(\sqrt{-c}R)}}$

2. $\frac{du}{dt}=2\frac{d^2u}{dx^2}$

$u(t,x)=A(t)B(x)$

$\frac{dA}{dt}B(x)=2A(t)\frac{d^2B}{dx^2}$

$\frac{1}{A(t)}\frac{dA}{dt}=\frac{2}{B(x)}\frac{d^2B}{dx^2}=\lambda$

$\frac{dA}{dt} = \lambda A \Rightarrow A = A_0 e^{\lambda t}$

$\frac{d^2B}{dx^2} = \frac{\lambda}{2} B$

(a) $\frac{du}{dx}(t,0) = 0$ & $\frac{du}{dx}(t,10) = 0$

$c>0: h(x) = \alpha e^{\sqrt{\frac{\lambda}{2}}x} + \beta e^{-\sqrt{\frac{\lambda}{2}}x}$

$h'(x) = \sqrt{\frac{\lambda}{2}}\left(\alpha e^{\sqrt{\frac{\lambda}{2}}x} - \beta e^{-\sqrt{\frac{\lambda}{2}}x}\right)$

$h'(0) = \sqrt{\frac{\lambda}{2}}(\alpha - \beta) = 0$

$\alpha = \beta$

$h'(10) = \sqrt{\frac{\lambda}{2}}\alpha\left(e^{\sqrt{\frac{\lambda}{2}}10} - e^{-\sqrt{\frac{\lambda}{2}}10}\right) = 0$

$\alpha = 0 \quad \beta = 0$

$c=0: h(x) = \alpha + \beta x$

$h'(x) = \beta$

$h'(0) = h'(10) = \beta = 0$

$\alpha \in \mathbb{R}$

$\boxed{u(t,x) = A_0 e^{\lambda t}\alpha = A_0\alpha}$

$c<0: h(x) = \alpha\cos\left(\sqrt{-\frac{\lambda}{2}}x\right) + \beta\sin\left(\sqrt{-\frac{\lambda}{2}}x\right)$

$h'(x) = \sqrt{-\frac{\lambda}{2}}\left(-\alpha\sin\left(\sqrt{-\frac{\lambda}{2}}x\right) + \beta\cos\left(\sqrt{-\frac{\lambda}{2}}x\right)\right)$

$h'(0) = \sqrt{-\frac{\lambda}{2}}(\beta) = 0$

$\beta = 0$

$h'(10) = \sqrt{-\frac{\lambda}{2}}\left(-\alpha\sin\left(\sqrt{-\frac{\lambda}{2}}10\right)\right) = 0$

$\alpha \in \mathbb{R}, \sqrt{-\frac{\lambda}{2}}10 = n\pi \quad \boxed{\lambda = -2\left(\frac{n\pi}{10}\right)^2}$

$\boxed{u(t,x) = A_0 e^{\lambda t}\alpha\cos\left(\frac{n\pi}{10}x\right)}$

(b) $\frac{du}{dx}(t,0) = 0$ & $u(t,10) = 0$

$c>0: h(x) = \alpha e^{\sqrt{\frac{\lambda}{2}}x} + \beta e^{-\sqrt{\frac{\lambda}{2}}x}$

$h'(x) = \sqrt{\frac{\lambda}{2}}\left(\alpha e^{\sqrt{\frac{\lambda}{2}}x} + \beta e^{-\sqrt{\frac{\lambda}{2}}x}\right)$

$h'(0) = \sqrt{\frac{\lambda}{2}}(\alpha + \beta) = 0$

$\alpha = -\beta$

$h(10) = \alpha\left(e^{\sqrt{\frac{\lambda}{2}}10} - e^{-\sqrt{\frac{\lambda}{2}}10}\right) = 0$

$\alpha = 0 \quad \beta = 0$

$c=0: h(x) = \alpha + \beta x$

$h'(x) = \beta$

$h'(0) = \beta = 0$

$h(10) = \alpha = 0$

$c<0: h(x) = \alpha\cos\left(\sqrt{-\frac{\lambda}{2}}x\right) + \beta\sin\left(\sqrt{-\frac{\lambda}{2}}x\right)$

$h'(x) = \sqrt{-\frac{\lambda}{2}}\left[-\alpha\sin\left(\sqrt{-\frac{\lambda}{2}}x\right) + \beta\cos\left(\sqrt{-\frac{\lambda}{2}}x\right)\right]$

$h'(0) = \sqrt{-\frac{\lambda}{2}}\beta = 0 \quad \beta = 0$

$h(10) = \alpha\cos\left(\sqrt{-\frac{\lambda}{2}}10\right) = 0$

$\boxed{\alpha \in \mathbb{R}, \sqrt{-\frac{\lambda}{2}}10 = \left(n+\frac{1}{2}\right)\pi} \quad \boxed{\lambda = \left(\frac{(n+\frac{1}{2})\pi}{10}\right)^2(-2)}$

$\boxed{u(t,x) = A_0 e^{\lambda t}\alpha\cos\left(\frac{n+\frac{1}{2}}{10}\pi x\right)}$

(c) $u(t,0) = 0$ & $\frac{du}{dx}(t,10) = 0$

$c>0: h(x) = \alpha e^{\sqrt{\frac{\lambda}{2}}x} + \beta e^{-\sqrt{\frac{\lambda}{2}}x}$

$h(0) = \alpha + \beta = 0 \quad \alpha = -\beta$

$h'(10) = \sqrt{\frac{\lambda}{2}}\alpha\left(e^{\sqrt{\frac{\lambda}{2}}10} - e^{-\sqrt{\frac{\lambda}{2}}10}\right) = 0$

$\alpha = 0 \quad \beta = 0$

$c=0: h(x) = \alpha + \beta x$

$h(0) = \alpha = 0$

$h'(10) = \beta = 0$

$c<0: h(x) = \alpha\cos\left(\sqrt{-\frac{\lambda}{2}}x\right) + \beta\sin\left(\sqrt{-\frac{\lambda}{2}}x\right)$

$h(0) = \alpha = 0$

$h'(10) = \sqrt{-\frac{\lambda}{2}}\beta\cos\left(\sqrt{-\frac{\lambda}{2}}10\right) = 0$

$\boxed{\beta \in \mathbb{R}, \sqrt{-\frac{\lambda}{2}}10 = \left(n+\frac{1}{2}\right)\pi} \quad \boxed{\lambda = \left(\frac{(n+\frac{1}{2})\pi}{10}\right)^2(-2)}$

$\boxed{u(t,x) = A_0 e^{\lambda t}\beta\sin\left(\frac{n+\frac{1}{2}}{10}\pi x\right)}$

3. $\frac{du}{dt} = -c\frac{du}{dx} + ru$

$u(t,x) = A(t)B(x)$

$\frac{dA}{dt}B(x) = -cA(t)\frac{dB}{dx} + rA(t)B(x)$

$\frac{1}{A(t)}\frac{dA}{dt} = -\frac{c}{B(x)}\frac{dB}{dx} + r = \lambda$

$\frac{dA}{dt} = \lambda A \quad A = A_0 e^{\lambda t}$

$-\frac{c}{B}\frac{dB}{dt} = \lambda - r$

$\frac{dB}{dt} = -\frac{\lambda - r}{c}B = \frac{r-\lambda}{c}B$

$B = B_0 e^{\frac{r-\lambda}{c}x}$

$u(t,x) = A_0 e^{\lambda t} B_0 e^{\frac{r-\lambda}{c}x}$