Math 19. Lecture 33
Causes of Chaos

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1 The Lorenz Equations

We first look at a what seems a simple system called the Lorenz equations. This set of equations was devised to model certain weather-related phenomena. The system can be written as

$$\begin{aligned}\frac{dx}{dt} &= -\sigma x + \sigma y,\\ \frac{dy}{dt} &= rx - y - xz,\\ \frac{dz}{dt} &= -bz + xy.\end{aligned}$$

It was discovered that for certain parameters, the trajectories of the solutions were incredibly convoluted and effectively unpredictable. Here, σ, r, and b are constants. For certain values of these constants, the trajectories are both crazy and extremely sensitive to their starting positions.

In general, a system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, z),\\ \frac{dy}{dt} &= g(x, y, z),\\ \frac{dz}{dt} &= h(x, y, z),\end{aligned}$$

or in vector form

$$\frac{dv}{dt} = f(v).$$

has a unique solution for each initial condition.

- The solution can stay in a bounded region of the three dimensional version of the phase plane and wind through the region along an incredibly convoluted path.
- The solution may be very sensitive to initial data. Since real data always has some inherent uncertainty, starting values are never precisely known.
- No matter how long you watch a trajectory, you may not be able to predict future behavior.
- There is still value, but you must take care.

2 Equilibrium Points When v Has Two Components

The situation is fairly straightforward.

3 Equilibrium Points When v Has Three Components

The situation is much more complicated in three dimensions.

4 Throwing the Dice

Two trajectories that start close ending up far away.

5 Unpredictability for Two-Component Systems

This sort of unpredictability can only occur once along a trajectory (once for each hyperbolic equilibrium point).

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 28.