Math 19. Lecture 32
Testing for Periodicity

T. Judson

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1 Testing for a Term with Period p

Certain phenomena may be the sum of periodic functions. For example,

$$m(t) = m_d(t) + m_m(t) + m_y(t)$$

might be a function depending on a daily cycle (m_d), and monthly cycle (m_m), and a yearly cycle (m_y). Yet, $m(t)$ itself may or may not look periodic.

- Suppose that we know $m(t)$ on the interval $[-\tau_0, -\tau_0 + \tau]$. Here τ_0 represents the amount of time before the present at which our data begins and τ is the time span for which we have collected data.

- We wish to know if our data $m(t)$ has a periodic component of period p. If p is much less than τ, we may be able to detect such a period.

- We must compute the following three integrals:

$$a \equiv \frac{1}{\tau}\int_{-\tau_0}^{-\tau_0+\tau} m(t)\cos(2\pi t/p)\, dt \tag{1}$$

$$b \equiv \frac{1}{\tau}\int_{-\tau_0}^{-\tau_0+\tau} m(t)\sin(2\pi t/p)\, dt \tag{2}$$

$$\sigma \equiv \frac{1}{\tau}\int_{-\tau_0}^{-\tau_0+\tau} m(t)^2\, dt. \tag{3}$$

- With these numbers computed, we then compute

$$f = \frac{a^2 + b^2}{\sigma}, \tag{4}$$

which has a value between 0 and 1. A significant component of the original function $m(t)$ with period p is signified by a large value for f.

2 The Power Spectrum Function

In general, we may not have an a priori guess of what the periods of the components of m might be (if there are periods). In practice, a standard approach is to compute many values of f in (4) and see which values of p give us large f.

3 Fourier Coefficients

It is customary to change variables from the period p to the frequency $\nu = 1/p$. Then (1) and (2) become

$$a \equiv \frac{1}{\tau}\int_{-\tau_0}^{-\tau_0+\tau} m(t)\cos(2\pi\nu t)\ dt \tag{5}$$

$$b \equiv \frac{1}{\tau}\int_{-\tau_0}^{-\tau_0+\tau} m(t)\sin(2\pi\nu t)\ dt. \tag{6}$$

In this case, f becomes a function of ν:

$$f(\nu) = \frac{a(\nu)^2 + b(\nu)^2}{\sigma}.$$

This function is called the *power spectral density* function.

4 An Example

Suppose that

$$m(t) = \alpha\cos(2\pi t/q) + \beta\sin(2\pi t/q).$$

is periodic with period q. We compute the Fourier coefficients and the power spectral density function.

5 Trigonometric Integrals

Fourier coefficients can often be computed in exact form. We need the following indefinite integrals. If $A \neq B$, then

- $\displaystyle\int \cos(At)\cos(Bt)\ dt = \frac{\sin((A-B)t)}{2(A-B)} + \frac{\sin((A+B)t)}{2(A+B)}$
- $\displaystyle\int \sin(At)\cos(Bt)\ dt = \frac{\cos((A-B)t)}{2(A-B)} - \frac{\cos((A+B)t)}{2(A+B)}$
- $\displaystyle\int \sin(At)\sin(Bt)\ dt = \frac{\sin((A-B)t)}{2(A-B)} - \frac{\sin((A+B)t)}{2(A+B)}$

If $A = B$, then

- $\displaystyle\int \cos^2(At)\ dt = \frac{t}{2} + \frac{1}{4A}\sin(2At)$
- $\displaystyle\int \sin(At)\cos(At)\ dt = -\frac{1}{4A}\cos(2At)$
- $\displaystyle\int \sin^2(At)\ dt = \frac{t}{2} - \frac{1}{4A}\sin(2At)$

6 Example

Suppose that $m(t) = e^{-|t|}$ is defined on $(-\infty, \infty)$.

- $$\begin{aligned} a(\nu) &= \frac{1}{\tau}\int_{-\infty}^{\infty} e^{-|t|}\cos(2\pi\nu t)\ dt \\ &= \frac{2}{\tau}\int_{0}^{\infty} e^{-t}\cos(2\pi\nu t)\ dt \\ &= \frac{2}{\tau}\left[\frac{2\pi\nu e^{-t}\sin(2\pi\nu t) - e^{-t}\cos(2\pi\nu t)}{1+4\pi^2\nu^2}\right]_0^{\infty} \\ &= \frac{2}{\tau(1+4\pi^2\nu^2)} \end{aligned}$$

- $$b(\nu) = \frac{1}{\tau}\int_{-\infty}^{\infty} e^{-|t|}\sin(2\pi\nu t)\,dt = 0.$$

- $$\sigma = \frac{1}{\tau}\int_{-\infty}^{\infty} e^{-2|t|}\,dt = \frac{2}{\tau}\int_{0}^{\infty} e^{-2t}\,dt = \frac{2}{\tau}\left[-\frac{e^{-2t}}{2}\right]_0^{\infty} = \frac{1}{\tau}.$$

- $$f(\nu) = \frac{a(\nu)^2 + b(\nu)^2}{\sigma} = \frac{4}{\tau(1+4\pi^2\nu^2)^2}$$

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 27.