Math 19. Lecture 30
Estimating Elapsed Time

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1 Estimating Time

Suppose that

$$\frac{ds}{dt} = f(s), \tag{1}$$

where $s(0) = s_0$. If $s_0 \leq s \leq s_1$ and $f(s) > 0$, we wish to estimate how long it takes to reach s_1.

- *Step 1.* Let $t = t_1$ be the time where s first takes on the value s_1. Suppose that
  - $f_{\max}$ be the maximum value of $f(s)$ on $[s_0, s_1]$.
  - $f_{\min}$ be the minimum value of $f(s)$ on $[s_0, s_1]$.
- *Step 2.* From equation (1),

  $$f_{\min} \leq \frac{ds}{dt} \leq f_{\max} \tag{2}$$

  on $[s_0, s_1]$.
- *Step 3.* Integrating (2) from 0 to t_1 and using the Fundamental Theorem of Calculus, we have

  $$f_{\min} \cdot t_1 \leq \int_0^{t_1} \frac{ds}{dt}\, dt \leq f_{\max} \cdot t_1$$

  or

  $$f_{\min} \cdot t_1 \leq s(t_1) - s(0) = s_1 - s_0 \leq f_{\max} \cdot t_1.$$

- *Step 4.* Therefore,
$$\frac{s_1 - s_0}{f_{\max}} \leq t_1 \leq \frac{s_1 - s_0}{f_{\min}}.$$

2 Some Examples

- Let
$$\frac{dx}{dt} = 2 + \sin(\pi x)$$
with initial condition $x(0) = 0$. We wish to find upper and lower bounds for t when $x(t) = 1$. We can replace $\sin \pi x$ by 1 to get a maximum for $2 + \sin(\pi x)$ and by 0 to get a minimum for $2 + \sin(\pi x)$. Therefore,
$$\frac{1}{3} \leq t \leq \frac{1}{2}.$$

- Let
$$\frac{dx}{dt} = 2x^4 - x + 2$$
with initial condition $x(0) = 0$. We wish to find upper and lower bounds for t when $x(t) = 1$. The function $2x^4 - x + 2$ has a critical point at $x = 1/2$. Since
$$\begin{aligned} f(0) &= 2 \\ f(1/2) &= 13/8 \\ f(1) &= 3, \end{aligned}$$
we know that
$$\frac{1}{3} \leq t \leq \frac{8}{13}.$$

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 25.