Math 19. Lecture 23
Stability Criterion (I)

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1 Linear Stability Criterion

Let u_e be an equilibrium solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0. \tag{2}$$

The solution $u_e(x)$ is a stable solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$

$$\frac{d}{dx} u_e(0) = \frac{d}{dx} u_e(L) = 0$$

if and only if there is *no* pair (g, λ) , where $g(x)$ is some function that is *not* identically zero for $0 \leq x \leq L$, where $\lambda \in \mathbb{R}$, and where the following constraints are satisfied.

- $\lambda \geq 0$
- $\lambda g = \mu \frac{d^2}{dx^2} + z(x) g$
- $\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0$

A solution is unstable if there is even one such pair (g, λ) that obeys the above conditions.

2 Some Heuristic Justifications

1. The solutions to (1) or (2) that are close to the equilibrium solution $u_e(x)$ can be written as

$$u(t, x) = u_e(x) + w(t, x),$$

   where $|w|$ is small when near x and t under consideration.

2. If $|w|$ is small, then

$$\frac{f(u_e + w) - f(u_e)}{w} \approx \left.\frac{df}{du}\right|_{u=u_e} = z(x).$$

   So we can replace $f(u_e(x) + w(x))$ by

$$f(u_e) + \left.\frac{df}{du}\right|_{u=u_e} w = f(u_e) + z(x)w. \tag{3}$$

   Substituting this replacement expression into (1), we get

$$\frac{\partial}{\partial t}(u_e + w) = \mu \frac{\partial^2}{\partial x^2}(u_e + w) + f(u_e) + z(x)w. \tag{4}$$

   There is nothing mysterious going on here. We are just estimating $f(u)$ with its first-order Taylor series representation.

3. Since u_e is independent of time and obeys (3), equation (4) becomes

$$\frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2} + z(x)w. \tag{5}$$

   The boundary conditions for w are now

$$\frac{\partial w}{\partial x}(t, 0) = \frac{\partial w}{\partial x}(t, L) = 0. \tag{6}$$

   To write

$$z(x) = \left.\frac{df}{du}\right|_{u=u_e},$$

   we have to know u_e. Equation (5) is much simpler than equation (1), because w appears as a first power and not as some potentially complicated function like $f(u_e + w)$.

4. We can now solve (5) and (6) using the separation of variables technique. Let $w(t,x) = A(t)g(x)$. Then

$$g(x)\frac{dA}{dt} = \mu A(t)\frac{d^2g}{dx^2} + A(t)g(x)z(x)$$

or

$$\frac{1}{A}\frac{dA}{dt} = \frac{\mu}{g}\frac{d^2g}{dx^2} + z(x).$$

This gives us two equations

$$\frac{dA}{dt} = \lambda A \tag{7}$$

$$\lambda g(x) = \mu\frac{d^2g}{dx^2} + g(x)z(x). \tag{8}$$

The solution to (7) is $A = A(0)e^{\lambda t}$. If there exists a $\lambda \geq 0$ and a $g(x) \not\equiv 0$ such that

$$\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0,$$

then there exists a solution $w(t,x) = A(0)e^{\lambda t}g(x)$ such that $|w| \not\to 0$ as $t \to \infty$. Therefore, we have an unstable solution.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 19.
- "Direct and Continuous Assessment by Cells of Their Position in a Morphogen Gradient," pp. 296–300.
- "Activin Signalling and Response to a Morphogen Gradient," pp. 300–309.