Math 19. Lecture 18
No-Trawling Zones

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1 No Trawling Zones

Deep trawling can have a devastating effect on a fishery. If we were to implement no trawling zones, what should be the minimum effective width of such a strip? We will use lobsters as an indicator species. Suppose that lobster populations are destroyed outside of our no trawling zone by deep trawling. Inside our no trawling zone, the lobster population increases exponentially, say

$$\frac{du}{dt} = ru.$$

We will designate infinitely long strips of width R as no trawl zones. Our goal is to estimate how wide the strip should be so that the lobster population will not decrease. The lobster population should obey the equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + ru.$$

The first term on the right is the diffusion term accounting for lobsters randomly roaming the bottom of the ocean. The second term on the right is the population growth due to reproduction. We will assume that this growth is exponential. The constant μ is the diffusion constant. Both μ and r may be estimated by lab experiments or field observations.

2 Boundary Conditions

We should look for solutions such that

$$u(t,0) = u(t,R) = 0.$$

We will show that the lobster population will grow with time provided

$$R > \left(\frac{\mu\pi^2}{r}\right)^{1/2}.$$

Hence, R depends on μ and r as we might expect.

3 Separation of Variables

Let us assume that we can find a solution of the form

$$u(t,x) = A(t)B(x).$$

For $u(t,x) = A(t)B(x)$ to be a solution to

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + ru,$$

it must be the case that

$$B(x)\frac{dA}{dt} = \mu A(t)\frac{d^2B}{dx^2} + rA(t)B(x),$$

where

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \left[\frac{dA}{dt}(t)\right]B(x), \\
\frac{\partial^2 u}{\partial x^2} &= A(t)\left[\frac{d^2B}{dx^2}(x)\right].
\end{aligned}$$

Separating the variables in this last equation, we obtain

$$\frac{1}{A(t)}\frac{dA}{dt} = \frac{\mu}{B(x)}\frac{d^2B}{dx^2} + r.$$

The variables x and t are independent. For a function of time to be equal to a function of space, they must both be constant. Let

$$\frac{1}{A(t)}\frac{dA}{dt} = \frac{\mu}{B(x)}\frac{d^2B}{dx^2} + r = \lambda.$$

Thus, we obtain two ordinary differential equations,

$$\frac{dA}{dt} - \lambda A = 0, \quad (1)$$

$$\frac{d^2B}{dx^2} - \frac{\lambda - r}{\mu}B = 0. \quad (2)$$

The solution to the first equation is

$$A(t) = A(0)e^{\lambda t}.$$

Thus, $A(t)$ grows if $\lambda > 0$ and decays if $\lambda < 0$. Since we are interested in our lobster population surviving, we will assume that $\lambda \geq 0$.

We can rewrite equation (2) as

$$\frac{d^2B}{dx^2} - cB = 0,$$

where

$$c = \frac{\lambda - r}{\mu}.$$

We will consider three cases, $c > 0$, $c = 0$, and $c < 0$. The only nontrivial case occurs when $c < 0$. In this case,

$$B(x) = \alpha \cos \sqrt{-c}\, x + \beta \sin \sqrt{-c}\, x.$$

The boundary condition $B(0) = 0$ implies that $\alpha = 0$. The second boundary condition tells us that

$$\beta \sin\left(\sqrt{-c}\, R\right) = 0.$$

Therefore, either $\beta = 0$ (no interesting solutions) or

$$\sin\left(\sqrt{-c}\, R\right) = 0.$$

The sine function vanishes at multiples of π; hence, the later case is equivalent to

$$\sqrt{-c}\, R = n\pi,$$

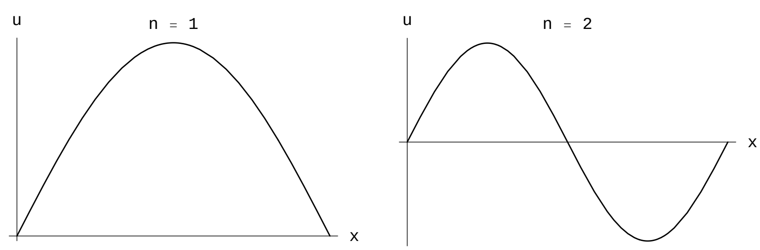
where n is any integer. Since μ, r, and R are fixed, we are restricted on how we may choose the value of λ. Our solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + ru.$$

is

$$u(t,x) = A(t)B(x) = \beta e^{\lambda t} \sin\left(\frac{n\pi x}{R}\right),$$

where $0 < x < R$ and $\beta > 0$. Since our solution must be positive, $n = 1$.



Therefore, our solution is

$$u(t,x) = A(t)B(x) = \beta e^{\lambda t} \sin\left(\frac{\pi x}{R}\right).$$

Since $c = (\lambda - r)/\mu$ and $\sqrt{-c}\, R = \pi$,

$$\lambda = r - \frac{\mu\pi^2}{R^2}.$$

For the lobster population to grow,

$$0 < \lambda = r - \frac{\mu\pi^2}{R^2}$$

or

$$R > \left(\frac{\mu\pi^2}{r}\right)^{1/2}.$$

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 16.