Math 19. Lecture 16
The Diffusion Equation

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Fall 2005

1 Advection—One More Time

Recall the advection equation

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} + k(t,x),$$

where

- $q(t,x)$ is the *net* number of particles that pass x from left to right.
- $k(t,x)$ is the *net* number of particles created per unit length at time t and position x. *What comes in either stays in or goes out again.*

The advection equation becomes useful when we can specify q and k. In our example, we had

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} - ru$$

with solution

$$u(t,x) = e^{-rt} f(x - ct),$$

where f is any differentiable function evaluated at $s = x - ct$.

To verify the solution, notice that

$$\begin{aligned} \frac{\partial u}{\partial t} &= -re^{-rt} f(x-ct) + e^{-rt}\frac{\partial}{\partial t} f(x-ct), \\ \frac{\partial u}{\partial x} &= e^{-rt}\frac{\partial}{\partial x} f(x-ct). \end{aligned}$$

If

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} - ru,$$

then

$$\begin{aligned}\frac{\partial u}{\partial t} &= -re^{-rt}f(x-ct) + e^{-rt}\frac{\partial}{\partial t}f(x-ct) \\ &= -c\frac{\partial u}{\partial x} - ru \\ &= -ce^{-rt}\frac{\partial}{\partial x}f(x-ct) - re^{-rt}f(x-ct),\end{aligned}$$

or

$$\frac{\partial}{\partial t}f(x-ct) = -c\frac{\partial}{\partial x}f(x-ct).$$

Applying the chain rule, it must be the case that

$$-cf(x-ct) = -cf(x-ct).$$

This equation works when particle motions is entirely due to advection.

2 The Diffusion Equation

If the ambient fluid is a rest, then particle movement can be modeled by the *diffusion* or *heat equation*,

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + k(t,x),$$

where k may depend on u. The constant μ is called the *diffusion coefficient*, where

$$\mu = \sqrt{\text{average of the squares of the velocities of the particles.}}$$

The diffusion equation is

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} + k(t,x),$$

where

$$q(t,x) = -\mu\frac{\partial u}{\partial x}$$

is the net number of particles moving from left to right per unit time, which is proportional to the change in density.

3 Solutions to the Diffusion Equation

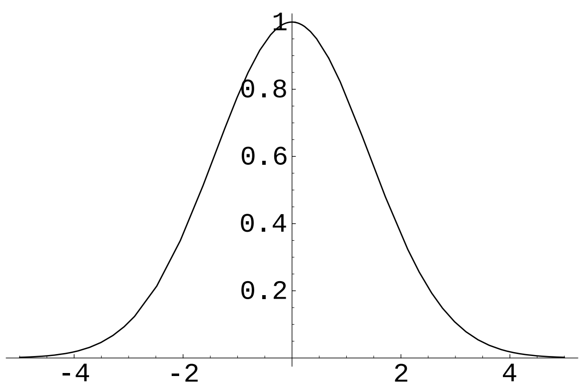
Solutions to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

are of the form

$$u(t,x) = \frac{a}{\sqrt{t}} e^{-x^2/4\mu t}.$$

For a fixed t, these solutions graph as normal curves.



Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 14.
- "Past Temperatures Directly from the Greenland Ice Sheet," pp. 238–245.