Math 19. Lecture 14
Introduction to Advection (I)

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1 An Explosion Example

Suppose a meltdown at a nuclear reactor pumps radioactive pollution into the air. A wind blows from west to east at 3 m/sec. Particles fall out of the air at a constant rate r. We wish to know the particle concentration east and west of the explosion at location x and time t. Let the density function,

$$u = u(t, x),$$

be the number of particles per meter.

The amount of particulate mater in the region between x and $x + \Delta x$ at time t is approximately $u(t, x)\Delta x$. The rate of change with respect to time is

$$\frac{d}{dt}u(t, x)\Delta x = q(t, x) - q(t, x + \Delta x) + k(t, x)\Delta x,$$

where

- $q(t, x)$ is the number of particles that pass x from left to right, so $-q(t, x)$ is the number of particles that pass x from right to left.
- $q(t, x + \Delta x)$ is the number of particles that pass $x + \Delta x$ from left to right, so $-q(t, x+\Delta x)$ is the number of particles that pass $x+\Delta x$ from right to left.
- $k(t, x)$ is the net number of particles created in $[x, x + \Delta x]$. That is, $k(t, x)$ is the number created minus the number destroyed. In our example,

$$k(t, x) = -ru(t, x).$$

Thus,

$$\frac{d}{dt}u(t,x)\Delta x = q(t,x) - q(t,x+\Delta x) + k(t,x)\Delta x,$$

or

$$\frac{d}{dt}u(t,x) = -\frac{q(t,x+\Delta x) - q(t,x)}{\Delta x} + k(t,x).$$

As $\Delta x \to 0$, this last expression becomes

$$\frac{\partial u}{\partial t}(t,x) = -\frac{\partial q}{\partial x}(t,x) + k(t,x).$$

In our example,

$$\begin{aligned} k(t,x) &= -ru(t,x) \\ q(t,u) &= 3u(t,x). \end{aligned}$$

Thus, we obtain the *advection equation.*

$$\frac{\partial u}{\partial t} = -3\frac{\partial u}{\partial x} - ru.$$

2 Solutions to the Advection Equation

Every solution to

$$\frac{\partial u}{\partial t} = -3\frac{\partial u}{\partial x} - ru.$$

can be written in the form

$$u(t,x) = e^{-rt}f(x-3t),$$

where f is any differentiable function in one variable. The choice of f is determined by initial and boundary conditions.

Homework

- Chapter 13. Exercises 1, 3, 5, 6, 7, 8; pp. 213–215.

Reading and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 13.
- "Malaria: Focus on Mosquito Genes" pp. 198–202.