Math 19. Lecture 4
Introduction to Differential Equations

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1 Modeling a Population with a Carrying Capacity

Suppose that we are to analyze the population of fish in a lake and that we know from past experience that the maximum number of fish that the lake will support is 200 (thousand?) fish. How can we model this population?

2 The Logistic Equation

The logistic equation is commonly used to model population growth in a resource limited environment. It was first used by the Belgian biologist Verhulst to predict the populations of Belgium and France. We can adjust the model of exponential growth, $dp/dt = kp$, to account for limited environment and limited resources. We make the following assumptions.

- If the population is small, the rate of growth of the population is proportional to its size.

$$\frac{dp}{dt} = kp \text{ if } p \text{ is small.}$$

- If the population is too large to be supported by its environment and resources ($N =$ carrying capacity), the population will decrease. If $p > N$, then

$$\frac{dp}{dt} < 0.$$

- The correct equation might be of the form

$$\frac{dp}{dt} = kh(p)p$$

The function $h(p)$ should be close to 1 if p is small, but negative if $p > N$. The simplest expression that has these properties is

$$h(p) = \left(1 - \frac{p}{N}\right).$$

- The *logistic population model* is

$$\frac{dp}{dt} = k\left(1 - \frac{p}{N}\right)p \text{ or } \frac{dp}{dt} = kp - \frac{1}{N}p^2.$$

To get the equation in the book, let $b = 1/N$.

- *This equation is a mathematical model. It can only be verified by experiment!*

3 dfield

dfield is the program that will solve differential equations. A java versions of dfield can be found at `http://math.rice.edu/~dfield/`

4 Some Vocabulary

- An *equilibrium solution* is constant for all values of the independent variable. The graph is a horizontal line. Equilibrium values can identified by setting the derivative of the function equal to zero.

- An equilibrium solution is *stable* if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends towards positive infinity.

- An equilibrium solution is *unstable* if a small change in the initial conditions gives a solution which veers away from the equilibrium as the independent variable tends towards positive infinity.

5 The Effects of Harvesting

Now look at effect of different levels of fishing on a fish population. If fishing takes place at a continuous rate of H fish per year, the fish population P satisfies the differential equation

$$\frac{dP}{dt} = 2P - 0.01P^2 - H.$$

6 Some Important Points

- The qualitative behavior of solutions $p(t)$ to the logistic, and to the generic equation
$$\frac{dp}{dt} = f(p),$$
can be obtained from information in the graph of the function f.

- An equation of the form
$$\frac{dp}{dt} = f(p),$$
is completely predictive. Choose any starting value for p and there is precisely one solution that starts at your chosen starting time with your chosen starting value.

- The points where the graph crosses the p-axis correspond to the constant solutions to the differential equation. Meanwhile, if $p(t)$ is a time dependent solution and $f(p(t)) > 0$, then $p(t)$ moves to the right on the p-axis as time increases. Conversely, if $f(p(t)) < 0$, then $p(t)$ moves to the left.

7 Global Temperature

Let $T = T(t)$ be the average temperature of the earth's surface at time t. This can be modeled by
$$\frac{dT}{dt} = f_{\text{in}} - f_{\text{out}}.$$

- f_{in} is the contribution from incoming solar energy. f_{in} increases with T since a greater fraction of the earth's surface is covered with ice at lower temperatures. The ice reeflects sunlight and reduces the radiation reaching the surface of the earth.

- f_{out} is the contribution from the outgoing radiation. f_{out} increases with T since a hot body radiates more than a cool one.

Homework

- Chapter 3. Exercises 1, 2, 3, 4; pp. 62–63.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 3.