Math 19. Lecture 3
Exponential Growth

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Fall 2005

1 Predicting the Population of the U.S.

Given the following census data of the U.S. population, how can we predict the population in 2010?

Year	t	Actual	$P(t)$	Year	t	Actual	$P(t)$
1790	0	3.9	3.9	1910	120	91	47
1800	10	5.3	4.8	1920	130	105	58
1810	20	7.2	5.9	1930	140	122	72
1820	30	9.6	7.3	1940	150	131	88
1830	40	12	9	1950	160	151	108
1840	50	17	11	1960	170	179	133
1850	60	23	14	1970	180	203	164
1860	70	31	17	1980	190	226	202
1870	80	38	21	1990	200	249	249
1880	90	50	25	2000	210	281	306
1890	100	62	31	2010	220	—	377
1900	110	75	38	2020	230	—	464

2 The Exponential Equation

Consider a population of $P(t)$ at time t. During each unit of time, say Δt, a constant fraction of population will be having offspring. We will also assume that the population has a constant death rate. Thus, the change in the population during the interval Δt is

$$\Delta P \approx k_{\text{birth}} P(t) \Delta t - k_{\text{death}} P(t) \Delta t$$

where k_{birth} is the fraction of the population having children during the interval and k_{death} is the fraction of the population that dies during the interval. Therefore,

$$\frac{\Delta P}{\Delta t} \approx kP(t),$$

where $k = k_{\text{birth}} - k_{\text{death}}$. Since the derivative of P is

$$\frac{dP}{dt} = \lim_{\Delta \to \infty} \frac{\Delta P}{\Delta t},$$

the rate of change of the population is proportional to the size of the population,

$$\frac{dP}{dt} = kP$$

at time t.

3 The Population of the U.S. Revisited

Consider the U.S. population model:

$$\begin{aligned} \frac{dP}{dt} &= kP \\ P(0) &= 3.9 \\ P(200) &= 249 \end{aligned}$$

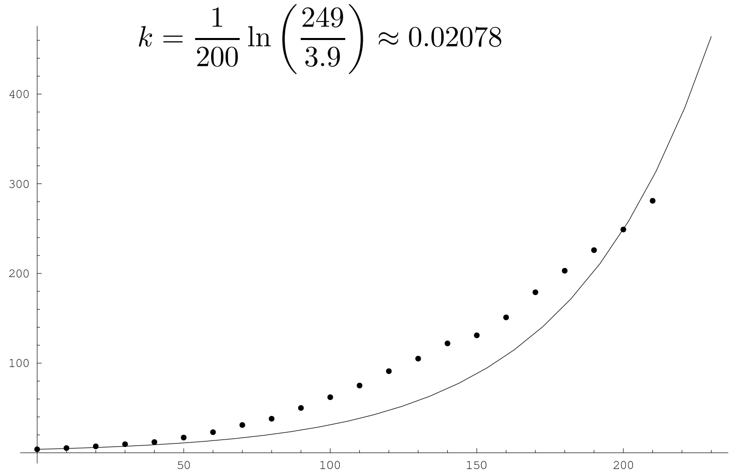
The general solution is

$$P(t) = P(0)e^{kt}.$$

In the example,

$$P(t) = 3.9e^{kt}$$

$$k = \frac{1}{200} \ln\left(\frac{249}{3.9}\right) \approx 0.02078$$



4 Bufo Marinis—What Can Go Wrong

The American marine toad (*Bufo marinis*) was introduced into Australia to control sugar cane beetles. Unfortunately, the toads are nocturnal feeders and the beetles are abroad by day. The following table provides the land area in Australia colonized by the toad from 1939–1974.

Year	Cumulative area occupied (km^2)
1939	32,800
1944	55,800
1949	73,600
1954	138,000
1959	202,000
1964	257,000
1969	301,000
1974	584,000

5 The Equation $\frac{dq}{dt} = aq + c$

The equation

$$\frac{dq}{dt} = aq + c$$

has solution

$$q(t) = \left(q(0) + \frac{c}{a}\right) e^{at} - \frac{c}{a}$$

We can show this two ways: directly and by making the substitution $p(t) = q(t) + c/a$. The latter reduces the equation to the exponential growth equation.

6 Sums of Exponential Functions

Beware of sums of exponential functions of time such as

$$f(t) = e^{-t} + e^{-4t},$$

where $t \geq 0$. This might be the level of the AIDS virus in a patient's blood predicted for t dates after the beginning of a particular drug therapy. The term in the sum with the least negative or most positive exponential will dominate the sum for large t.

7 Taylor's Theorem

The second reason why $dP/dt = aP$ occurs so often has to do with Taylor's theorem. Any function $f(x)$ can be approximated near a point x_0 by an nth degree polynomial

$$g_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f(n)(x_0)(x - x_0)^n.$$

Homework

- Chapter 2. Exercises 1, 2, 4 (a, b, c, d), 5; pp. 43–44.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 2.
- "HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time," pp. 15–22.